

# ESTIMATING LOBLOLLY PINE SIZE-DENSITY TRAJECTORIES ACROSS A RANGE OF PLANTING DENSITIES

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**Abstract**—Size-density trajectories on the logarithmic (ln) scale are generally thought to consist of two major stages. The first is often referred to as the density-independent mortality stage where the probability of mortality is independent of stand density; in the second, often referred to as the density-dependent mortality or self-thinning stage, the probability of mortality is related to stand density. Within the self-thinning stage, segments of a size-density trajectory consisting of a nonlinear approach to a linear portion, a linear portion (maximum size-density relationship dynamic thinning line), and a divergence from the linear portion are generally assumed. Estimates of the ln of quadratic mean diameter and ln of trees per acre where the two stages of stand development and the three phases of self-thinning begin and end were obtained from segmented regression analyses and used as response variables predicted as a function of planting density. Predicted values allow for size-density trajectories to be estimated for any planting density.

## INTRODUCTION

Self-thinning quantifies the relationship between average tree size and tree density and has been widely studied. Understanding self-thinning is important to better grasp intraspecific mortality patterns of a tree species growing in even-aged stands leading to more efficient management of growing stock. For instance, estimating the onset of self-thinning can help resource managers plan thinnings and reduce competition-induced mortality. Quantifying maximum size-density relationships (MSDR), or the maximum obtainable tree density per unit area for a given quadratic mean diameter ( $D$ ), should help resource managers better understand how different management regimes affect productivity. Predictions of MSDRs can be used to constrain and verify estimated stand development of process-based models and those empirical models that were developed using data limited in ranges of density and/or age to properly estimate mortality equations. Maximum size-density relationships have been used as constraints in several growth-and-yield models (Monserud and others 2004, Poage and others 2007) both for the  $\ln V$ - $\ln N$  relationship (e.g., Landsberg and Waring 1997, Smith and Hann 1984, Turnbull and Burk 2000) and the  $\ln N$ - $\ln D$  relationship (e.g., Hynynen 1993, Johnson 2000). In many model systems, mortality equations are combined with height, diameter, or volume equations to estimate an approach to a linear MSDR constraint, and once the projected stand density is equivalent to the linear constraint, self-thinning occurs such that stand density is maintained equivalent to the linear constraint for some period of time.

VanderSchaaf (2006) and VanderSchaaf and Burkhart (2008) proposed using segmented regression to provide a less subjective, statistically based criteria to determine what observations are within various stages and phases of stand development and at what  $\ln N$  and  $\ln D$  the various stages and phases begin and terminate, where  $\ln$  is the natural logarithm and  $N$  is trees per acre. In this paper, equations are presented to predict size-density trajectories of loblolly

pine (*Pinus taeda* L.) plantations across a range of planting densities using results obtained from VanderSchaaf (2006) and VanderSchaaf and Burkhart (2008).

## Stages of Stand Development and Phases of Self-Thinning

For size-density trajectories on the  $\ln$  scale, two major stages of stand development are generally recognized (Drew and Flewelling 1979, McCarter and Long 1986, Williams 1994): the first being an initial stage without significant competition in which mortality is independent of stand density (fig. 1—stage I), often referred to as the density-independent mortality stage, and the second being a stage with competition-induced mortality (the self-thinning stage) often referred to as the density-dependent mortality stage (fig. 1—stage II). Within

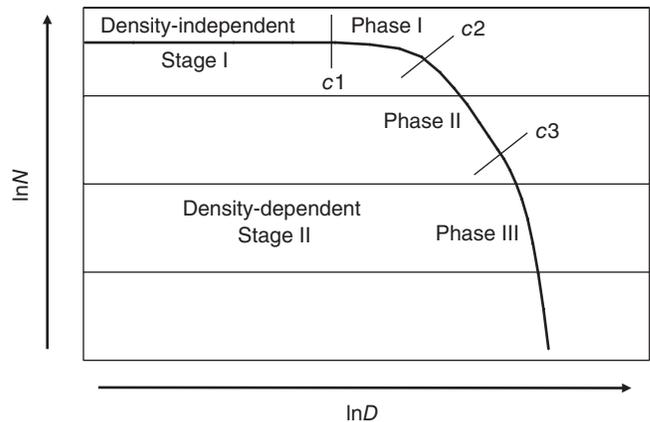


Figure 1—Depiction of a size-density trajectory for an individual stand. The two stages of stand development are shown—density-independent mortality and density-dependent mortality. Within the density-dependent mortality stage, or when self-thinning is occurring, three phases of stand development are shown. The join points ( $c_1$ ,  $c_2$ ,  $c_3$ ) used in equation (2) to differentiate stages and phases of stand development in size-density trajectories are depicted.

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the overall self-thinning stage, when density-dependent mortality is occurring, three phases are generally assumed. The first phase is represented by a nonlinear approach of a size-density trajectory, followed by a linear portion of a trajectory, and the third phase is represented by a divergence of the size-density trajectory from the linear portion. A further explanation is given below:

**Phase I**—Initially, the self-thinning stage of stand development can be represented by a curved approach of a size-density trajectory to a linear portion of self-thinning (or the MSDR dynamic thinning line) (fig. 1—phase I). During this initial component of self-thinning, mortality is less than the mortality at maximum competition and thus the trajectory has a concave shape (del Rio and others 2001, Harms and others 2000, Poage and others 2007).

**Phase II**—With increases in tree sizes and the death of other trees, eventually the size-density trajectory is assumed to become linear (fig. 1—phase II) where an increase in  $D$  (inches) is a function of the stand's maximum value of Reineke's (1933) stand density index (SDI), the change in  $N$ , and the MSDR dynamic thinning line slope ( $b$ ). Known as the MSDR dynamic thinning line phase of stand development (Weller 1990), this is when a stand is fully stocked (del Rio and others 2001) and Reineke's SDI remains relatively constant. Reineke's SDI is expressed as:

$$SDI = N(D/10)^b \quad (1)$$

where

SDI = Reineke's SDI

$N$  = trees per acre

$D$  = quadratic mean diameter (inches)

$b$  = exponent of Reineke's equation, equivalent to the MSDR dynamic thinning line slope on the ln-ln scale

**Phase III**—Eventually, as trees die, the residual trees cannot continue to fully occupy canopy gaps and the trajectory diverges (fig. 1—phase III) from the MSDR dynamic thinning line (Bredenkamp and Burkhart 1990, Cao and others 2000, Zeide 1995). The divergence from the MSDR dynamic thinning line has been depicted both as a line (Christensen and Peet 1981, Lonsdale 1990, Peet and Christensen 1980) and as a curve (Cao and others 2000, Zeide 1985). Whether the divergence can be depicted as linear or a curve is probably related to the amount of time since the occurrence of the MSDR dynamic thinning line phase (Cao and others 2000, Christensen and Peet 1981, Weller 1991). For example, in figure 1, the time period immediately after the MSDR dynamic thinning line phase of stand development shows an approximate linear divergence. With time, as mortality continues, the divergence becomes curvilinear eventually encompassing the disintegration portion of stand development.

Over the entire range of the density-dependent mortality stage of stand development the relationship between  $\ln N$  and  $\ln D$  is curvilinear; however, it is commonly assumed there is a linear phase (or portion) during self-thinning (Cao and others 2000, del Rio and others 2001, Hynynen 1993, Johnson 2000, Monserud

and others 2004, Poage and others 2007, VanderSchaaf and Burkhart 2008, Yang and Titus 2002, Zeide 1985).

## METHODS

### Data

Tree- and plot-level measurements were obtained from a spacing trial maintained by the Loblolly Pine Growth and Yield Research Cooperative at Virginia Polytechnic Institute and State University. The spacing trial was established on four cutover sites—two in the upper Atlantic Coastal Plain and two in the Piedmont. There is one Coastal Plain site in North Carolina and one in Virginia while both Piedmont sites are in Virginia. Three replicates of a compact factorial block design were established at each location in either 1983 or 1984. Sixteen initial planting configurations were established ranging in densities from 2,722 to 302  $N$ , a variety of planting distances between and within rows was used (not all spacings were square). Thus, a total of 192 experimental units were established when combining all 4 sites (4 sites by 3 replications by 16 planting configurations). For the planting densities of 2,722, 1,210, 680, and 302  $N$  there was 1 plot established for a particular site and replication combination; for the planting densities of 1,815, 1,361, 605, and 453  $N$  there were two plots established; and for the planting density of 907  $N$  there were 4 plots established. Seed sources were of genetically improved stock considered superior for a particular physiographic region; for both sites within a particular physiographic region the same genetic stock was used. All seedlings planted at each location were lifted from the same nursery and were 1-0 stock. See Sharma and others (2002) for a more comprehensive description of the studies.

Measurements of  $D$  and  $N$  were conducted annually between ages 5 and 21 on one of the Coastal Plain sites and to age 22 on the other site. On the Piedmont sites, measurement ages end at 18 at one location and 21 at the other. At the latter Piedmont site, one replication had measurements to 22 years of age. Site quality was quantified using site index defined as the average height of all trees with diameters larger than  $D$  for the planting densities of 907, 680, and 605  $N$  by replication. Plots intermediate in stand density were used when estimating site index for each replication in order to avoid any possible effects of high or low number of  $N$ . A site index equation found in Burkhart and others (2004) was used to project dominant height forward to base age 25. Table 1 contains summaries of plot-level characteristics for the entire dataset.

### Using Segmented Regression to Estimate Stages and Phases of Stand Development

A segmented regression model was developed based on the two stages of stand development and the three phases of self-thinning to objectively determine what observations of size-density trajectories are within particular stages and phases. The segmented regression model can be written as:

$$\ln N = (b_1)J_1 + (b_1 + b_2[\ln D - c_1]^2)J_2 + (b_1 + b_2[c_2 - c_1]^2 + b_3[\ln D - c_2])J_3 + (b_1 + b_2[c_2 - c_1]^2 + b_3[c_3 - c_2] + b_4[\ln D - c_3])J_4 \quad (2)$$

**Table 1—Plot-level characteristics for the entire dataset (n = 2977)**

Variable	Minimum	Mean	Maximum
Trees per acre	228	917	2,722
Quadratic mean diameter (inches)	1.1	5.4	10.8
Square feet of basal area per acre	0.1	122	258
Site index at base age 25 (feet)	63	68	73

where:

$D$  = quadratic mean diameter (inches), d.b.h. was measured at 4.5 feet above the ground

$J_1, J_2, J_3,$  and  $J_4$  = indicator variables for the stages and phases of stand development

$J_1$  = 1 if  $\ln D$  is within the density-independent mortality stage of stand development (stage I in fig. 1), zero otherwise

$J_2$  = 1 if  $\ln D$  is within the curved approach to the MSDR dynamic thinning line phase of self-thinning (phase I of stage II in fig. 1), zero otherwise

$J_3$  = 1 if  $\ln D$  is within the MSDR dynamic thinning line phase of self-thinning (phase II of stage II in fig. 1), zero otherwise

$J_4$  = 1 if  $\ln D$  is within the divergence phase of self-thinning (phase III of stage II in fig. 1), zero otherwise, and other variables as previously defined

Seven parameters were estimated for each planting density (VanderSchaaf 2006, VanderSchaaf and Burkhart 2008); one for the initial component where no density-related mortality occurs ( $b_1$ ), one for the curved approach to the MSDR dynamic thinning line ( $b_2$ ), one for the MSDR dynamic thinning line ( $b_3$ ), one for the divergence from the MSDR dynamic thinning line ( $b_4$ ), and three for the join points to estimate at what  $\ln D$  self-thinning begins ( $c_1$ ), at what  $\ln D$  the MSDR dynamic thinning line phase of stand development begins ( $c_2$ ), and at what  $\ln D$  the divergence from the MSDR dynamic thinning line begins ( $c_3$ ).

Convergence criteria were not met in parameter estimation of equation (2) for the planting densities of 453 and 302  $N$ . In previous reports (VanderSchaaf 2006, VanderSchaaf and Burkhart 2008), a system of simultaneously estimated equations were used to estimate at what  $\ln D$  and  $\ln N$  planting density-specific MSDR dynamic thinning lines begin and terminate. This paper extends the work of those publications by using a system of simultaneously estimated equations to also estimate at what  $\ln D$  self-thinning begins and the size-density trajectory coefficients of various stages and phases ( $b_2, b_3,$  and  $b_4$ ). Additionally, this work presents estimates of the  $N$  after density-independent (or random) mortality ( $b_1$ ). The seven values for each dependent variable for the planting densities ranging from 2,722 to 605  $N$  as estimated using segmented regression are presented in table 2.  $\ln N$

values by planting density were derived using the parameter estimates of the segmented regression models as presented in VanderSchaaf (2006) and shown in table 2.

### Model Forms and Parameter Estimation

Due to a limited number of observations for model fitting, making it difficult to estimate the cross-equation random error correlation matrix, parameters of two distinct simultaneous systems of linear regression equations were estimated. The first system was used to model the density-independent mortality stage of stand development (stage I in fig. 1), and the second system was used to model phases I and II of the self-thinning stage of stand development (phases I and II of stage II in fig. 1). Phase III, or the divergence phase of the self-thinning stage of stand development (phase III of stage II in fig. 1), was modeled separately.

A simultaneous parameter estimation method presented in Borders (1989) was used for the two sets of simultaneous equations. The density-independent mortality stage linear system of equations is:

$$b_1 = b_{01} + b_{11} \ln(N_0) \quad (3)$$

$$\ln D_s = b_{02} + b_{12} \ln(b_1) \quad (4)$$

The density-dependent mortality stage linear system of equations is:

$$b_2 = b_{03} + b_{13} \ln(N_0) \quad (5)$$

$$\ln D_b = b_{04} + b_{14} \ln(N_0) \quad (6)$$

$$\ln D_e = b_{05} + b_{15} \ln D_b \quad (7)$$

$$\ln N_b = b_{06} + b_{16} \ln D_b \quad (8)$$

$$\ln N_e = b_{07} + b_{17} \ln D_e \quad (9)$$

The equation form used to estimate the slope of the divergence phase of size-density trajectories is:

$$b_4 = b_{08} + b_{18} \ln(N_0) \quad (10)$$

where

$\ln D_s$  =  $\ln D$  corresponding to the initiation of the self-thinning stage of stand development (7  $c_1$  estimates from table 2)

$\ln D_b$  =  $\ln D$  corresponding to the initiation of a particular MSDR dynamic thinning line (7  $c_2$  estimates from table 2)

$\ln D_e$  =  $\ln D$  corresponding to the termination of a particular MSDR dynamic thinning line (7  $c_3$  estimates from table 2)

$\ln N_b$  =  $\ln N$  corresponding to the initiation of a particular MSDR dynamic thinning line

$\ln N_e$  =  $\ln N$  corresponding to the termination of a particular MSDR dynamic thinning line

$N_0$  = planting density (trees per acre)

$b_{0i}, b_{1i}$  = parameters to be estimated

**Table 2—Dependent variable values used in estimating parameters of equations (3) to (10) as obtained from segmented regression model results presented in VanderSchaaf (2006)**

Planting density	Stage I		Stage II						
	<i>b</i> 1	Curved approach		MSDR dynamic thinning line				Divergence	
		<i>lnDs</i> ( <i>c</i> 1)	<i>b</i> 2	<i>lnDb</i> ( <i>c</i> 2)	<i>lnNb</i>	<i>b</i> 3	<i>lnDe</i> ( <i>c</i> 3)	<i>lnNe</i>	<i>b</i> 4
<i>per acre</i>									
2,722	7.8833	1.1103	-1.8300	1.3737	7.7563	-1.8852	1.4855	7.5456	-3.7231
1,815	7.4773	1.2228	-1.3237	1.5691	7.3186	-1.6777	1.6649	7.1578	-3.4829
1,361	7.1886	1.3536	-1.3897	1.6535	7.0636	-1.1109	1.7335	6.9747	-2.7154
1,210	7.0648	1.3868	-1.1343	1.7104	6.9460	-1.4331	1.8228	6.7849	-4.3940
907	6.7691	1.5551	-1.0541	1.8382	6.6846	-1.7074	1.8940	6.5893	-1.9898
680	6.5001	1.5554	-0.5454	1.9674	6.4075	-1.4385	2.0994	6.2176	-13.7855
605	6.3635	1.6532	-0.5607	2.0320	6.2830	-1.6226	2.0908	6.1876	-2.2319

The system of equations will avoid illogical predictions of the response variables, e.g., *lnDb* estimated to be greater than *lnDe*. Using a *ln* transformation of planting density to predict *b*1, *b*2, *lnDb*, and *b*4 allows for a nonlinear relationship between these variables. For equation (10), the divergence slope for the 680 N planting density was removed. Hence, for equation (10), *n* = 6 and for all other equations, *n* = 7. Parameter estimates (SAS 1989) are given in table 3.

Rather than directly modeling the MSDR dynamic thinning line slope (*b*3), an alternative formula as shown in VanderSchaaf (2006) and VanderSchaaf and Burkhart (2008) was used:

$$b3 = (\ln Nb - \ln Ne) / (\ln Db - \ln De) \quad (11)$$

This helps to reduce the number of dependent variables in the simultaneous estimation equation system.

## RESULTS AND DISCUSSION

All parameter estimates were significant at an alpha = 0.05 level except for equations (3) and (10). When excluding the 680 planting density observation, there is a slight trend between *b*4 and planting density. For the purposes of this paper, when depicting size-density trajectories (fig. 2), a divergence from the MSDR dynamic thinning line (phase III of stage II—fig. 1) was included as estimated using equation (10).

Based on equations (3) through (11) and the data used in fitting those equations and the original segmented regression models (VanderSchaaf 2006, VanderSchaaf and Burkhart 2008), the MSDR boundary level differs relative to planting density (fig. 2). Although the MSDR dynamic thinning lines appear to be short in duration, the level of

intraspecific competition among trees when a stand's size-density trajectory is within the linear phase of self-thinning can be quite intense and a stand may stay in this phase for many years. It should be realized that the predicted size-density trajectories depicted in figure 2 provide no information about the rate of change in *D* across time. Rates of change in *D* can be relatively large during the density-independent mortality stage and the divergence phase of self-thinning.

When using equation (3), which estimates the *N* at which self-thinning begins, predicted percent survival ranged from 96.5 to 97.5 percent and varied little relative to planting density, as expected. Initial survival depends on seedling care, planting practices or depth, localized interspecific competition, diseases or infestations from the nursery, etc., and how these factors interact with local climatic conditions. Predicted survival rates from these spacing trials are not necessarily indicative of those that might be realized in operational plantings.

Due to divergences from each individual stand's linear boundary, the predicted size-density trajectories suggest that for the trajectories of the stands (plots) used in model fitting, e.g., loblolly pine plantations in the Atlantic Coastal Plain and Piedmont regions, the maximum *lnN* for a given *lnD* across all planting densities can be a conglomeration from several stands. VanderSchaaf and Burkhart (2007) noted this behavior may lead to a MSDR species boundary line slope that is not representative of how, on average, individual stands self-thin during the linear phase of the density-dependent mortality stage. For instance, for the four size-density trajectories presented in figure 2, a step interval

**Table 3—Parameter estimates of equations (3) and (4) and (5) through (9) that were simultaneously estimated and equation (10)**

Equation	Intercept			Slope			RMSE	Adj. $R^2$
	Estimate	Std. error	Sign.	Estimate	Std. error	Sign.		
(3)	-0.07677	0.0408	0.1184	1.006456	0.00576	<0.0001	0.0344	0.9691
(4)	6.319165	0.3584	<0.0001	-2.52196	0.1838	<0.0001	0.00757	0.9998
(5)	4.748849	0.6430	0.0007	-0.83049	0.0907	0.0003	0.1293	0.9208
(6)	4.769806	0.0717	<0.0001	-0.42948	0.0101	<0.0001	0.0150	0.9958
(7)	0.156933	0.0608	0.0494	0.962787	0.0344	<0.0001	0.0302	0.9818
(8)	10.81729	0.0843	<0.0001	-2.24479	0.0481	<0.0001	0.0305	0.9965
(9)	10.84589	0.0806	<0.0001	-2.22531	0.0437	<0.0001	0.0290	0.9966
(10)	4.910022	4.9455	0.3770	-1.1177	0.6894	0.1803	0.8094	0.2456

Std. error is the standard error of the estimate; RMSE = root mean squared error.

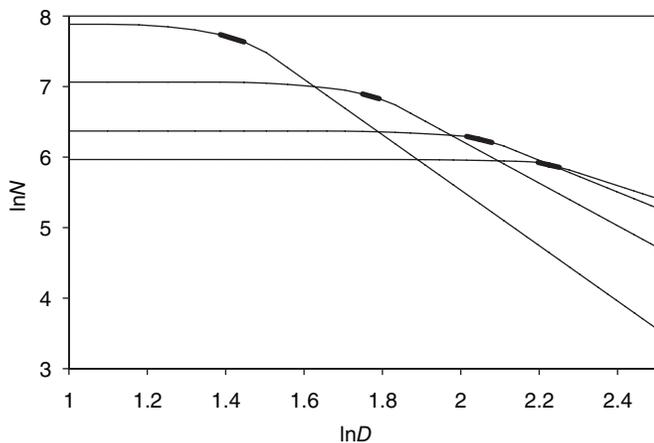


Figure 2—Predicted size-density trajectories using equations (3) through (11) for planting densities of 405 (beyond the planting density range of the model fitting dataset), 605, 1,205, and 2,722 seedlings per acre.

was used to create observations from the MSDR dynamic thinning line portion. Combining all created observations together into a single linear regression model resulted in  $\ln N = 10.80946 - 2.21884 \ln D$ . This MSDR species boundary line slope [defined by VanderSchaaf and Burkhardt (2007) as the MSDR species boundary line  $l$  slope] is not representative of the slopes of the individual size-density trajectories as estimated using the density-dependent mortality stage system of equations with an average of  $-1.48316$ . VanderSchaaf and Burkhardt (2010) noted similar behavior for stands differing in planting densities and site qualities.

These predicted size-density trajectories can be used to help determine rates of density-independent mortality, when self-thinning is expected to occur, and as constraints or verifications of both empirical and process-based model size-density trajectories.

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